Weekly Assignment 3 Solutions (Due Tuesday 7/13 at 11:59PM)

Overview: This assignment is graded out of **96 points**. There are **7 problems**, each of which is worth **16 points** for a total of **112 points** possible. **Therefore, a score above 96 points earns you extra credit.** Your score for each question will depend on the graders determination of your proficiency in each of the following categories: Conceptual Understanding, Strategies & Reasoning, Computation & Execution, and Communication. You can earn up to 4 points for each category. The grader determines your score for each category using the Weekly Assignment Rubric.

Guidelines: You are required to adhere to the weekly assignment guidelines and the assignment submission guidelines in the syllabus. If you fail to follow the guidelines, you risk receiving no credit for you work. Turn in your assignment via gradescope.

Directions: Complete the following exercises from the Active Calculus textbook. You can click the links below to go directly to the exercise.

1. Exercise 10.1.14.

Solution. **(a)** Take any function whose limit does not exact and define it to be anything at the origin - the limit does not "see" the function value. E.g.

$$p(x,y) = \begin{cases} \frac{x^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}.$$

The limit along the x axis is 0 but along the line $y = x$ it is $1/2$.

(b) Let $q(x,y) = \frac{x^2y}{x^4+y^2}$. Approaching the origin along any line of the form $y = mx$, we see that

$$\lim_{x\to 0} q(x, mx) = \lim_{x\to 0} \frac{mx^3}{x^4 + m^2x^2} = 0$$

since this is a rational function whose denominator is of larger degree than the numerator. But approaching the origin along a parabola of the form $y = mx^2$, we see that

$$\lim_{x\to 0} q(x, mx^2) = \lim_{x\to 0} \frac{mx^4}{(1+m^2)x^4} = \frac{1}{1+m^2}$$

for any $m \in \mathbb{R}$. Therefore, the limit at the origin does not exist, but yet the limit is the same along any line through the origin.

(c) This is not possible, since the definition of continuity of $r(x,y)$ at $(0,0)$ requires that the limit $\lim_{(x,y)\to(0,0)} r(x,y)$ exists.

(d) This clearly violates the uniqueness of the limit. If the limit exists, we must have the same value along any path. Therefore, no such function exists.

(e) Let $t(x,y) = \frac{(x-1)^2+(y-1)^2}{\sqrt{(x-1)^2+(y-1)^2+1}-1}$. Perform the computation and you will see that the limit $\lim_{(x,y)\to(1,1)} t(x,y)$ exists, but yet the function is not defined at $(1,1)$. Compare with Reading 5 Task 5(b).

□

2. Exercise 10.2.13.

Solutions. **(a)** According to Definition 10.2.4,

$$I_T(94, 75) = \lim_{h \to 0} \frac{I(94 + h, 75) - I(94, 75)}{h}.$$

Apply the symmetric difference quotient to an appropriate trace with $h = 2$ to estimate the partial derivative:

$$I_T(94, 75) \approx \frac{I(94 + 2, 75) - I(94 - 2, 75)}{2 \cdot 2} = \frac{130 - 115}{4} = 15/4.$$

The partial derivative $I_T(94, 75)$ says that if the actual temperature outside is 94 degrees Fahrenheit and the relative humidity is 75%, then it will feel 14/5 degrees Fahrenheit hotter for every 1 degree Fahrenheit increase in actual temperature.

(b) For the limit definition, apply Definition 10.2.4. The estimation is similar to (a) but with $h = 5$. We have

$$I_H(94, 75) \approx \frac{I(94, 75 + 5) - I(94, 75 - 5)}{2 \cdot 5} = \frac{127 - 118}{10} = 9/10.$$

The partial derivative $I_H(94, 75)$ says that if the actual temperature outside is 94 degrees Fahrenheit and the relative humidity is 75%, then it will feel 9/10 degrees Fahrenheit hotter for every 1 percent increase in humidity.

(c) We use a linear approximation to to the appropriate traces. If $g(x)$ is a function, then $g(a + h) \approx g(a) + g'(a)h$ for small h. We replace f with a trace and use an appropriate value of h. Using $h = -1$, we have

$$I(91, 80) = I(92 - 1, 80) \approx I(92, 80) - I_T(92, 80) = 119 - 3.75 = 115.25.$$

Similarly, with $h = -2$,

$$I(92, 78) = I(92, 80 - 2) \approx I(92, 80) - 2I_H(92, 80) = 119 - 1.6 = 117.4.$$

(d) The average rate of change with respect to time is given by the change in the heat index by the change in time. Therefore, the average rate of change from 1pm to 3pm is given by

$$\frac{I(96, 75) - I(92, 85))}{2} = \frac{130 - 123}{2} = 3.5.$$

The units are apparent degrees Fahrenheit per hour.

□

3. Exercise 10.3.12.

Solutions. **(a)** According to Definition 10.2.4,

$$I_{TT}(94, 75) = \lim_{h \to 0} \frac{I_T(94 + h, 75) - I_T(94, 75)}{h}.$$

Apply the symmetric difference quotient to an appropriate trace with $h = 2$ to estimate the second-order partial derivative:

$$I_{TT}(94, 75) \approx \frac{I_T(94 + 2, 75) - I(94 - 2, 75)}{4}.$$

We must estimate $I_T(96, 75)$ and $I_T(92, 75)$. For the latter, use the same method as in the 10.2.13 to get $I_T(92, 75) \approx 13/4$. For the former, the symmetric difference quotient would require knowledge of the value $I(98, 75)$, which is not in the table. So instead, we approximate $I_T(96, 75)$ us a secant line as follows:

$$I_T(96, 75) \approx \frac{I(96, 75) - I(94, 75)}{2} = \frac{130 - 122}{2} = 4.$$

Therefore,

$$I_{TT}(94, 75) \approx \frac{4 - 13/4}{4} = 3/16.$$

The partial derivative $I_{TT}(94, 75)$ tells us that if it is 94 degrees Fahrenheit outside and the humidity is 75%, I_T increases by 3/16 for each 1 degree Fahrenheit increase in actual temperature.

(b) For the limit definition, apply Definition 10.2.4. The estimation is similar to (a) but with $h = 5$. We have

$$I_{HH}(94, 75) \approx \frac{I_H(94, 75 + 5) - I_H(94, 75 - 5)}{2 \cdot 5} = \frac{1 - 4/5}{10} = 1/50.$$

The partial derivatives $I_H(94, 80) \approx 1$ and $I_H(94, 70) \approx 4/5$ were estimates as usual with a symmetric difference quotient, and a secant line, respectively. The partial derivative $I_{HH}(94, 75)$ says that if the actual temperature outside is 94 degrees Fahrenheit and the relative humidity is 75%, then I_H will increase by 1/50 apparent degrees Fahrenheit for every 1 percent increase in relative humidity.

(c) Setting up the limit is easy. To approximate the partial derivative, we have

$$I_{HT}(94, 75) \approx \frac{I_H(94 + 2, 75) - I_H(94 - 2, 75)}{4} = \frac{1 - 7/10}{4} = 3/40.$$

The partials $I_H(94 + 2, 75) \approx 1$ and $I_H(94 - 2, 75) \approx 7/10$ were approximated using symmetric difference quotients with $h = 5$, as usual. The values tells us that if the actual temperature outside is 94 degrees Fahrenheit and the relative humidity is 75%, then I_H will increase by 3/40 apparent degrees Fahrenheit for every 1 degree Fahrenheit increase in actual temperature.

$\square$

4. Let f be a function that is continuously differentiable at $(2, 7)$ and suppose that its tangent plane at this point is given by

$$z = -1 + 4(x - 2) - 3(y - 7).$$

(a) Determine the values of $f(2, 7)$, $f_x(2, 7)$, and $f_y(2, 7)$. Explain your reasoning for each part.

(b) Estimate the value of $f(1.8, 7.2)$. Show your work and explain your reasoning.

(c) Given the changes $dx = -0.23$ and $dy = .12$, estimate the corresponding change in f that is given by the differential df.

(d) Suppose that g is another function continuously differentiable at $(2, 7)$ with tangent plane given by

$$x - y - z = 10.$$

Determine $g(2, 7)$, $g_x(2, 7)$, and $g_y(2, 7)$ and then estimate $g(1.8, 7.2)$. Explain your reasoning.

Solution. **(a)** Since the plane is tangent to the graph of f at $(2,7)$, $f(2,7) = z(2,7) = -1$. Since the plane is a tangent plane, $f_x(2,7) = z_x(2,7) = 4$ and $f_y(2,7) = z_y(2,7) = -3$. The reasoning is that the tangent plane and the graph have the same tangent lines in the x and y directions.

(b) The linearization is $L(x,y) = -1 + 4(x-2) - 3(y-7)$ and therefore $f(1.8, 7.2) \approx L(1.8, 7.2) = -2.4$.

(c) We know that $df = f_x(2,7)dx + f_y(2,7)dy$. Therefore,

$$df = 4 \cdot (-0.23) + (-3) \cdot (.12) = -1.28.$$

(d) The plane needs to be written in the appropriate form in order to read the desired information. We have

$$\begin{aligned} z &= -10 + x - y = -10 + (x-2) - (y-7) + 2 - 7 \\ &= -15 + (x-2) - (y-7). \end{aligned}$$

Now, we can read that $g(2,7) = z(2,7) = -15$, $g_x(2,7) = z_x(2,7) = 1$, and $g_y(2,7) = z_y(2,7) = -1$. Using $L(x,y) = -15 + (x-2) - (y-7)$, we have $g(1.8, 7.2) \approx L(1.8, 7.2) = -15.4$.

□

5. Let g be the function defined by the equation $g(x,y) = 4x^3 + 2y^2$.

(a) Find the equation of the tangent plane to g at the point $(1,1)$.

(b) Use the local linearization to approximate the values of g at the points $(1.1, 1.05)$ and $(1.3, 1.2)$.

(c) Compare your approximations with the exact values $g(1.1, 1.05)$ and $g(1.3, 1.2)$. Which approximation is the most accurate? Why should this be expected?

Solution. **(a)** The equation of the tangent line is, according to the definition, given by

$$z = 6 + 12(x-1) + 4(y-1).$$

(b) The linearization is given by $L(x,y) = 6 + 12(x-1) + 4(y-1)$. Therefore,

$$g(1.1, 1.05) \approx L(1.1, 1.05) = 7.4$$

and

$$g(1.3, 1.2) \approx L(1.3, 1.2) = 10.4.$$

(c) The actual values are $g(1.1, 1.05) = 7.53$ and $g(1.3, 1.2) = 11.67$. The second approximation is the least accurate. This is expected because the approximating function is linear and the point $(1.3, 1.2)$ is further away from $(1,1)$ that $(1.1, 1.05)$ is.

□

6. Exercise 10.5.12.

Solution. **(a)** Clearly $u(0, 2\pi/3) = \cos 2\pi/3 = -1/2$ and $v(0, 2\pi/3) = \sin 2\pi/3 = \sqrt{3}/2$.

(b) Use a tree diagram to determine the appropriate formulae. They are given by

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\ &= 2ue^x \cos y - 2ve^x \sin y\end{aligned}$$

and

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \\ &= -2ue^x \sin y - 2ve^x \cos y.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial z}{\partial x}(0, 2\pi/3) &= 2u(0, 2\pi/3)e^0 \cos 2\pi/3 - 2v(0, 2\pi/3)e^0 \sin 2\pi/3 \\ &= -1/2 - 9/2 = -5.\end{aligned}$$

The other computation is similar.

□

7. Exercise 10.6.11.

Solution. **(a)** We have

$$E_x(x, y) = -\frac{200(x-5)}{(1+(x-5)^2+4(y-2.5)^2)^2}$$

and

$$E_y(x, y) = -\frac{800(y-2.5)}{(1+(x-5)^2+4(y-2.5)^2)^2}.$$

(b) The unit vector is $u = \frac{\langle -3,4 \rangle}{5} = \langle -4/5, 3/5 \rangle$. Then

$$D_u f(3, 4) = \nabla f(3, 4) \cdot \langle -4/5, 3/5 \rangle = \langle 400/196, -1200/196 \rangle \cdot \langle -4/5, 3/5 \rangle \approx -5.3.$$

Since the function models a landmass, suppose we are standing on that landmass at the coordinates $(3, 4)$. For every 1 meter we walk in the direction of u, our elevation will decrease by 5.3 meters.

(c) The direction of greatest increase is given by $\nabla f(3, 4) = \langle 400/196, -1200/196 \rangle$.

(d) The rate of change in the direction of greatest increase is given by

$$|\nabla f(3, 4)| = \sqrt{(400/196)^2 + (-1200/196)^2} = \frac{100\sqrt{10}}{49} \approx 6.45.$$

Therefore, the instantaneous rate of change in the direction of greatest decrease is $-|\nabla f(3, 4)| = -6.45$. In other words, if we are standing at $(3, 4)$ and facing in the steepest downhill direction, our elevation will decrease by 6.45 meters for every 1 meter we walk in that direction.

(e) We seek a unit vector w such that $D_w f(3, 4) = 0$. Since $w = \langle a, b = \rangle$ is a unit vector, we have $a^2 + b^2 = 1$ and the other equation yields $\frac{400}{196}a - \frac{1200}{196}b = 0$. Solving this system yields two solutions, namely $w = \langle a, b \rangle = \langle 3/\sqrt{10}, 1/\sqrt{10} \rangle$ and $-w$.

□